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The Kosterlitz–Thouless Phase Transition in Two-Dimensional Hierarchical Coulomb Gases

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A hierarchical version of two-dimensional lattice Coulomb gases is investigated. For $\beta > \beta_c = 8\pi$ there is a locally stable line of fixed points for the renormalization group ("block charges") transformations. For $\beta > \bar{\beta}_c$ ($\beta_c \leqslant \bar{\beta}_c \leqslant \frac{3}{2}\pi\beta_c$), these fixed points are globally stable. As a consequence we show that there is no screening of external charges for any activity if $\beta > \bar{\beta}_c$. At β_c a supercritical bifurcation takes place and we investigate the behavior of the model for $\beta \lesssim \beta_c$ to show a weak form of screening.

KEY WORDS: Kosterlitz–Thouless; Coulomb gas; hierarchical model; renormalization group; screening; stability; bifurcation.

1. INTRODUCTION

It is well known[1] that two-dimensional lattice Coulomb gases exhibit a Kosterlitz–Thouless phase transition, which is characterized by the existence of a critical temperature β_c such that for $\beta < \beta_c$ the systems display Debye screening of fractional external charges and for $\beta > \beta_c$ no screening takes place.

Already in the early attempts at understanding the phenomenon renormalization group (RG) techniques were recognized to be of relevance.[1,2] Even the rigorous proof, by Fröhlich and Spencer,[3] of the low-temperature behavior of the systems ($\beta \gg \beta_c$) makes strong use of renormalization group ideas.

For $\beta \ll \beta_c$, Debye screening was proved in ref. 13 using earlier results by Brydges and Federbush.[14] Practically nothing is rigorously known about the behavior of the systems around β_c.

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This paper is part of a program initiated in refs. 5 and 8 aiming at a fuller understanding of the Kosterlitz–Thouless scenery using RG techniques.

As usual,[6,7] our initial step is to consider a hierarchical version of the model. This is the content of the present work.

Our hierarchical model is defined by the replacement of the "lattice" Coulomb potential by

$$V_H(x, y) = -\frac{1}{2\pi} \ln d_L(x, y) \tag{1.1}$$

where $d_L(\cdot, \cdot)$ is the "hierarchical distance function" first introduced by Bleher and Sinai[9] and defined below. With this modified Coulomb potential we consider a family $\mathscr{F}$ of Coulomb gases with different *a priori* charge probability distributions λ which are mapped into each other by RG transformation $R: \mathscr{F} \to \mathscr{F}$:

$$\lambda \to \lambda' = R\lambda \tag{1.2}$$

corresponding to the formation of block charges. In contradistinction to refs. 3 and 8 we work directly with charge configurations, no use being made of the so-called sine-Gordon representation.[4]

The trivial fixed point λ_0 in this picture corresponds to the vacuum theory, i.e., the model with charge zero with *a priori* probability one. In this situation there is obviously no screening and our first result is a proof of the stability of this trivial fixed point for $\beta > \beta_c = 8\pi$. As a consequence, we are able to prove that for $\beta > \beta_c$ and λ close to λ_0 (in a suitably defined topology) there is no screening. The condition λ "close" to λ_0 amounts to a small activity condition. We also show that there exists $\bar{\beta}_c \geqslant \beta_c$ such that the results holds for all $\lambda \in \mathscr{F}$ (not necessarily close to λ_0!).

It is interesting to remark that for $\beta > \beta_c$ all directions in $\mathscr{F}$ around λ_0 are irrelevant and this feature simplifies very much the mathematical treatment of the problem.

At $\beta = \beta_c$ a supercritical bifurcation takes place: the trivial fixed point becomes unstable and a nontrivial line of fixed points appears with a different long-distance behavior of correlation functions. We prove that for $\varepsilon = (\beta_c - \beta)^{1/2}$ sufficiently small, these fixed points describe theories with a weak form of screening of Coulomb potential of external charges

$$V_H(x, y) \to -\frac{1}{C(\varepsilon)} \ln d_L(x, y) \tag{1.3}$$

where $C(\varepsilon) > 2\pi$ for $\varepsilon > 0$.

This screening for $\beta \lesssim \beta_c$ is not exponential and this is a feature of hierarchical models (see, for instance, ref. 6, Exercise 2, Chapter 4).

The bifurcation at $\beta = \beta_c$ is that of a simple eigenvalue and owing to the simplicity of the Banach space $\mathscr{F}$ where R acts, it is possible to apply the beautiful construction described in Crandall[11] of the bifurcating fixed point. The simplicity of our methods should be compared to the technical difficulties involved in the analogous $\varepsilon = 4 - d$ expansion for the $\lambda\phi^4$ hierarchical theory (see refs. 9, 10, and 12).

The hierarchical version of Coulomb gases we consider here has been independently proposed and discussed in refs. 5 and 8. Some of our results, namely those in the region $\beta > \beta_c$ and small activity for the standard gas (see Section 2 for definition), were also obtained in ref. 8 with a different approach. With our methods, based on the stability analysis of the trivial fixed point, we are able to obtain global stability for $\beta > \bar{\beta}_c$ and construct the nontrivial stable fixed point for $\beta \lesssim \beta_c$.

This paper is organized as follows. In Section 2 we introduce the family $\mathscr{F}$ of models to be considered. In Section 3 we describe the action of the RG transformation and prove the stability of the trivial fixed point for $\beta > \beta_c$. In Section 4 we discuss the asymptotic behavior of correlation functions for $\beta > \beta_c$ showing that they are governed by the behavior of the stable fixed point. In Section 5 we discuss the bifurcation and the screening properties of the nontrivial fixed point.

2. THE MODEL

A configuration q of the system enclosed in a finite volume $\Lambda \subset \mathbb{Z}^2$ is an assignment of an integer charge $q(x) \in \mathbb{Z}$ for each $x \in \Lambda$. The energy $E_\Lambda(q)$ of a configuration is given by

$$E_\Lambda(q) = \tfrac{1}{2}(q, V_H q) = \sum_{x,\, y \in \Lambda} q(x)\, V_H(x, y)\, q(y) \tag{2.1}$$

where

$$V_H(x, y) = -\frac{1}{2\pi} \ln d_L(x, y) \tag{2.2}$$

and $d_L(\cdot, \cdot)$ is the hierarchical distance function defined as follows.

Let $L > 1$ be an integer; for $x, y \in \mathbb{Z}^2$ we define

$$N_L(x, y) = \inf\{N \geqslant 1, N \text{ integer: } [L^{-N}x] = [L^{-N}y]\} \tag{2.3}$$

where for $z \in \mathbb{R}^2$, $[z] \in \mathbb{Z}^2$ and has as components the integer part of the components of z. We then set

$$d_L(x, y) = L^{N_L(x, y)} \tag{2.4}$$

Notice (a) the long-distance behavior

$$\lim_{|x-y|\to\infty} \frac{d_L(x, y)}{|x-y|} = 1 \tag{2.5}$$

and (b) the behavior under scaling

$$d_L(Lx, Ly) = \begin{cases} L & \text{if} \quad x = y \\ Ld_L(x, y) & \text{otherwise} \end{cases} \tag{2.6}$$

The unnormalized *a priori* probability distribution of charge configuration $F_\lambda(q)$ is determined by a charge activity function λ: $\mathbb{Z} \to \mathbb{R}$, $\lambda \geqslant 0$, through

$$F_\lambda(q) = \prod_{x \in \Lambda} \lambda(q) \tag{2.7}$$

where λ is going to be chosen in a suitably defined class $\mathscr{F}$ of acceptable functions, to be later specified.

Due to the usual infrared problems,[17] we shall consider only neutral configurations q, i.e., $\sum_{x \in \Lambda} q(x) = 0$. The Gibbs measure $\mu_{\beta,\lambda}^{(\Lambda)}$ at inverse temperature β is given by

$$\mu_{\beta,\lambda}^{(\Lambda)}(q) = \frac{F_\lambda(q)\, e^{-\beta E_\Lambda(q)}}{\Xi_{\beta,\lambda}^{(\Lambda)}} \tag{2.8}$$

with

$$\Xi_{\beta,\lambda}^{(\Lambda)} = \sum_{q: \sum q(x) = 0} F_\lambda(q)\, e^{-\beta E_\Lambda(q)} \tag{2.9}$$

The usual choices for λ are:

(a) Hard-core gas, with particle activity z:

$$\lambda_{\text{hc}}(q) = \delta_{q,0} + z(\delta_{q,1} + \delta_{q,-1}) \tag{2.10}$$

(b) Standard gas, with particle activity z:

$$\lambda_s(q) = I_n(2z) \tag{2.11}$$

where I_n denotes the nth modified Bessel function.

(c) Villain gas:

$$\lambda_V(q) = 1 \tag{2.12}$$

For this model the partition function is not well defined, since $\Xi^{(\Lambda)}_{\beta,\lambda_V} = \infty$, but we shall be able to provide a natural definition by a limiting procedure.

The reason why we consider λ belonging to a more general class $\mathscr{F}$ of models is due to the fact that block charge RG transformation as defined below will map $\lambda \to \lambda' \in \mathscr{F}$, with $\lambda' \neq \lambda$ in general.

3. RENORMALIZATION GROUP TRANSFORMATION AND THE TRIVIAL FIXED POINT

Our RG transformation involves, as usual,[6,15] two steps: integration over fluctuations and rescaling back to the original lattice. To that extent we introduce the rescaled block charge configuration q_B:

$$q_B(x) = \sum_{\substack{0 \leq y_i < L \\ i=1,2}} q(Lx + y) \tag{3.1}$$

The effective probability distribution of $\{q_B(x), x \in \mathbb{Z}^2\}$ may then be computed as follows.

We first notice that, due to the property (2.6),

$$\begin{aligned} E_{\Lambda_N}(q) &= \frac{1}{2} \sum_{x, y \in \Lambda_N} q(x)\, V_H(x, y)\, q(y) \\ &= E_{\Lambda_{N-1}}(q_B) - \frac{1}{4\pi} \ln L \sum_{\substack{x \neq y \\ x, y \in \Lambda_{N-1}}} q_B(x)\, q_B(y) \end{aligned} \tag{3.2}$$

where $\Lambda_N = \{[-L^N, L^N - L^{N-1}] \times [-L^N, L^N - L^{N-1}] \cap \mathbb{Z}^2\}$.

Neutrality then yields

$$E_{\Lambda_N}(q) = E_{\Lambda_{N-1}}(q_B) + \frac{1}{4\pi} \ln L \sum_{x \in \Lambda_{N-1}} [q_B(x)]^2 \tag{3.3}$$

The joint Gibbs probability distribution of $\{q_B(x), x \in \Lambda_{N-1}\}$ is then obtained from (2.8) after summation over q keeping q_B fixed. It is given by

$$\mu^{(\Lambda_{N-1})}_{\beta,\lambda'}(q_B) = \frac{1}{\Xi^{(\Lambda_{N-1})}_{\beta,\lambda'}} F_{\lambda'}(q_B) \exp[-\beta E_{\Lambda_{N-1}}(q_B)] \tag{3.4}$$

where

$$\lambda'(q) = L^{-(\beta/4\pi) q^2} \sum_{\substack{q_1, \dots, q_{L^2} \\ \sum q_i = q}} \lambda(q_1) \cdots \lambda(q_{L^2}) \tag{3.5}$$

Notice that apart from the volume change, $\Lambda_N \to \Lambda_{N-1}$, the joint probability distribution of $\{q_B(x), x \in \Lambda_{N-1}\}$ is again of the type (2.8) with λ replaced by λ'.

We are thus led to investigate the transformation $\lambda \to \lambda'$:

$$\lambda' = m_{\varphi_\beta}(\underbrace{\lambda * \lambda * \cdots * \lambda}_{L^2 \text{ times}}) \equiv m_{\varphi_\beta}(\lambda^{*L^2}) \tag{3.6}$$

where the convolution product $*$ in (3.6) is defined by

$$(f * g)(q) = \sum_{\substack{q_1, q_2 \in \mathbb{Z} \\ q_1 + q_2 = q}} f(q_1)\, g(q_2) \tag{3.7}$$

and m_{φ_β} is the multiplication operator by the function φ_β:

$$(m_{\varphi_\beta} f)(q) = \varphi_\beta(q)\, f(q) \tag{3.8a}$$

with

$$\varphi_\beta(q) = L^{-(\beta/4\pi) q^2} \tag{3.8b}$$

First of all we notice that the transformation $\lambda \to \lambda'$ is well defined if $\lambda \in l^1(\mathbb{Z})$, i.e., $\|\lambda\|_1 = \sum_{q \in \mathbb{Z}} |\lambda(q)| < \infty$. Moreover, if $\lambda \in l^1(\mathbb{Z})$, λ' is also in $l_1(\mathbb{Z})$ and, by Fubini's theorem,

$$\|\lambda'\|_1 \leqslant \|\varphi_\beta\|_\infty \, \|\lambda\|_1^{L^2} = (\|\lambda\|_1)^{L^2} \tag{3.9}$$

We are interested only in the subspace $l_e^1(\mathbb{Z})$ of even distribution $\lambda(q) = \lambda(-q)$. Notice that $\lambda' \in l_e^1(\mathbb{Z})$ if $\lambda \in l_e^1(\mathbb{Z})$. There is no loss of generality in considering only those $\lambda \in l_e^1(\mathbb{Z})$ such that $\lambda(0) \neq 0$, since this will be necessarily the case after the first iteration: if $\lambda \neq 0$, then

$$\lambda'(0) = \|\lambda^{*L^2/2}\|_2^2 > 0 \tag{3.10}$$

In order to avoid the annoying appearance of zero modes upon linearization of (3.6), we shall redefine the transformation by introducing a normalization factor $N(\lambda)$:

$$\lambda' = [1/\lambda^{*L^2}(0)]\, m_{\varphi_\beta}(\lambda^{*L^2}) = [1/N(\lambda)]\, m_{\varphi_\beta}(\lambda^{*L^2}) \tag{3.11}$$

with this choice, made possible by (3.10), we have

$$\lambda'(0) = 1 \tag{3.12}$$

Therefore there is no loss of generality in considering the *affine* subset of $l_e^1(\mathbb{Z})$, $\mathscr{F}$, given by

$$\mathscr{F} = \{\lambda \in l_e^1(\mathbb{Z}) \colon \lambda(0) = 1\} \tag{3.13}$$

From the above discussion it is clear that for the redefined map (3.11), $\lambda' \in \mathscr{F}$ if $\lambda \in \mathscr{F}$.

The transformation (3.11) has a trivial point $\lambda_0 \in \mathscr{F}$ given by

$$\lambda_0(q) = \delta_{q,0} = \begin{cases} 1, & q = 0 \\ 0, & q \neq 0 \end{cases} \tag{3.14}$$

Moreover, an arbitrary element $\lambda \in \mathscr{F}$ can be uniquely decomposed as

$$\lambda = \lambda_0 + \chi \tag{3.15}$$

where χ belongs to the *subspace* $\mathscr{G}$ of $l_e^1(\mathbb{Z})$:

$$\mathscr{G} = \{\chi \in l_e^1(\mathbb{Z}) \colon \chi(0) = 0\} \tag{3.16}$$

Therefore the transformation (3.11) maps $\lambda = \lambda_0 + \chi$, $\chi \in \mathscr{G}$, to $\lambda' = \lambda_0 + \chi'$ with $\chi' \in \mathscr{G}$ uniquely defined. For simplicity of analysis we shall often consider the "unphysical" case $L^2 = 2$, in which case the transformation $r \colon (\chi, \beta) \in \mathscr{G} \times \mathbb{R}_+ \to \chi' = r(\chi, \beta) \in \mathscr{G}$ is given explicitly by

$$\chi'(q) = \varphi_\beta(q) \frac{2\chi(q) + \sum_{q_1 + q_2 = q} \chi(q_1)\, \chi(q_2)}{1 + \sum_{q'} |\chi(q')|^2} \tag{3.17}$$

or

$$r(\chi, \beta) = \chi' = m_{\varphi_\beta} \frac{2\chi + \chi * \chi}{1 + \|\chi\|_2^2} \tag{3.18}$$

The trivial fixed point in $\mathscr{G}$ for the map r is the point 0. Its stability can be readily analyzed by considering the linear approximation $l \colon \mathscr{G} \times \mathbb{R}_+ \to \mathscr{G}$ of r around 0. It is given by a kernel $l_{qq'}$ of $l = dr(0, \beta)$,

$$(l\chi)(q) = \sum_{q' \in \mathbb{Z}} l_{qq'} \chi(q'), \qquad q \neq 0 \tag{3.19}$$

where

$$l_{qq'} = L^{2 - (\beta/4\pi) q^2} \delta_{q,q'} \tag{3.20}$$

The eigenvectors of $l(\beta) = dr(0, \beta)$ are $e_j(q) = (1/\sqrt{2})(\delta_{j,q} + \delta_{j,-q})$, $j = 1, 2, \ldots$, and the associated eigenvalues:

$$\omega_j(\beta) = L^{2 - (\beta/4\pi) j^2} \tag{3.21}$$

The spectrum of $l(\beta)$ is the closure of $\{\omega_j(\beta), j=1,2,...\}$ and so for $\beta>\beta_c=8\pi$, all eigenvalues (and the whole spectrum) are inside the unit circle in the complex plane.

We are therefore in a position to apply Lyapunov's linear stability Theorem[16] to conclude local stability of $\chi_0=0$ against small perturbation in the initial condition.

Theorem 3.1. Let $\beta>\beta_c=8\pi$; there is $\rho(\beta)>0$ and $\delta(\beta)<1$ such that

$$\|r^{(n)}(\chi,\beta)\|_1 \leqslant \delta^n \|\chi\|_1 \tag{3.22}$$

for $\chi\in\mathscr{G}$, $\|\chi\|_1<\rho(\beta)$.

Remark 3.2. For the hard-core gas, $\|\chi\|_1=z$, and for the standard gas, $\|\chi\|_1=2\sum_{q\geqslant 1}[I_q(2z)/I_0(2z)]=O(z^2)$ for $z\sim 0$. Therefore, for these gases the condition $\|\chi\|_1<\rho(\beta)$ is verified in the small-activity region.

For $\beta>\bar\beta_c$ a much stronger result holds: the trivial fixed point is globally stable.

Theorem 3.3. There exists $\bar\beta_c\geqslant\beta_c$, with $\bar\beta_c<\frac{3}{2}\pi\beta_c$, such that for $\beta>\bar\beta_c$,

$$\chi^{(n)}=r^{(n)}(\chi,\beta)\xrightarrow[n\to\infty]{} 0; \qquad \forall\chi\in\mathscr{G} \tag{3.23}$$

Proof. Let us for simplicity present the proof for $L^2=2$. From (3.18) it follows that

$$\begin{aligned}
\|\chi'\|_1 &\leqslant \frac{2\,\|\varphi_\beta\|_2\,\|\chi\|_2+\|\varphi_\beta\|_1\,\|\chi*\chi\|_\infty}{1+\|\chi\|_2^2}\\
&\leqslant \|\varphi_\beta\|_1\,\frac{2\,\|\chi\|_2+\|\chi\|_2^2}{1+\|\chi\|_2^2}\\
&\leqslant \|\varphi_\beta\|_1\sup_{a\geqslant 0}\frac{2a+a^2}{1+a^2}\leqslant 3\,\|\varphi_\beta\|_1\equiv R(\beta)
\end{aligned} \tag{3.24}$$

Therefore we may assume without loss of generality that $\|\chi\|_1\leqslant R(\beta)$. Inserting this again in (3.18), we obtain

$$\begin{aligned}
\|\chi'\|_1 &\leqslant 2\,\|\varphi_\beta\|_\infty\,\|\chi\|_1+\|\varphi_\beta\|_1\,\|\chi\|_1^2\\
&\leqslant (2\,\|\varphi_\beta\|_\infty+3\,\|\varphi_\beta\|_1^2)\,\|\chi\|_1\\
&<\left(\frac{1}{2}+\frac{6\pi^2}{\beta}\right)\|\chi\|_1
\end{aligned} \tag{3.25}$$

Remark 3.4. (a) The proof can be easily adapted to provide better upper bounds on $\bar{\beta}_c$. (b) For $\beta \gtrsim 8\pi$ the transformation $r(\chi, \beta)$ is not a strict contraction, as $dr(\chi, \beta)$ has an eigenvalue with absolute value bigger than one for χ not too far from 0.

Remark 3.5. The stability of the trivial fixed point for $\beta > 8\pi$ means from a physical point of view that the *a priori* probability of finding charged blocks goes to zero with the size the block going to infinity.

Remark 3.6. The Vilain model. This model can be considered as the limit $\kappa \to \infty$ of the model

$$\lambda_\kappa(q) = \begin{cases} 1, & |q| \leqslant \kappa \\ 0, & |q| > \kappa \end{cases} \tag{3.26}$$

$\lambda_\kappa \in \mathscr{F}$ and so we can construct λ'_κ as defined by (3.11), with the following properties:

$$\lambda'_\kappa(0) = 1; \qquad \lambda'_\kappa \in l^1(\mathbb{Z}); \qquad |\lambda'_\kappa(q)| \leqslant \varphi_\beta(q) \tag{3.27}$$

and $\lim_{\kappa \to \infty} \lambda'_\kappa(q) \equiv \lambda'_V(q)$ exists and is finite. Therefore λ'_V exists and belongs to $\mathscr{F}$.

Theorem 3.3 thus implies that for $\beta > \bar{\beta}_c$, $\lambda_V^{(n)} \to_{n \to \infty} \lambda_0$.

4. ABSENCE OF SCREENING FOR β > 8π

In this section we discuss how the electrostatic potential between two far apart external charges is affected by the presence of the gas. To do that, we consider for a given distribution γ of fractional external charges

$$\begin{aligned} &\gamma\colon\ \mathbb{Z}^2 \to (-1/2, 1/2) \\ &x \in \mathbb{Z}^2 \to \gamma(x) \in (-1/2, 1/2) \end{aligned} \tag{4.1}$$

the correlation function

$$G_\beta^{(\Lambda)}(\gamma) = \frac{\Xi_{\beta,\lambda}^{(\Lambda)}(\gamma)}{\Xi_{\beta,\lambda}^{(\Lambda)}(0)} \tag{4.2}$$

where $\Xi_{\beta,\lambda}^{(\Lambda)}(\gamma)$ is defined by the same expression (2.9) for $\Xi_{\beta,\lambda}^{(\Lambda)}(0) \equiv \Xi_{\beta,\lambda}^{(\Lambda)}$ with $E_\Lambda(q)$ replaced by

$$\begin{aligned} E_\Lambda(q+\gamma) &= \frac{1}{2} \sum_{x,y \in \Lambda} [q(x) + \gamma(x)]\, V_H(x,y) [q(y) + \gamma(y)] \\ &= E_\Lambda(\gamma) + E_\Lambda(q) + \sum_{x,y \in \Lambda} q(x)\, V_H(x,y)\, \gamma(y) \end{aligned} \tag{4.3}$$

We first notice the "easy" bound

$$G_\beta^{(\Lambda)}(\gamma) \geqslant e^{-\beta E_\Lambda(\gamma)} \tag{4.4}$$

This follows as in ref. 17 from (4.3), from $\lambda(q) = \lambda(-q)$ and Jensen's inequality.

Let us restrict our attention to the case when $\gamma(x)$ describes two external charges $+\gamma$ at x_0 and $-\gamma$ at y_0:

$$\gamma(x) = \begin{cases} +\gamma, & x = x_0 \\ -\gamma, & x = y_0 \\ 0, & \text{otherwise} \end{cases} \tag{4.5}$$

with $1 \leqslant |x_0 - y_0| < L^N$. In this case (4.4) implies, for any β,

$$-\frac{1}{\beta} \frac{\ln G_\beta^{(\Lambda)}(\gamma)}{\ln d_L(x_0, y_0)} \leqslant \frac{\gamma^2}{2\pi} \tag{4.6}$$

We shall now prove the opposite bound:

$$-\frac{1}{\beta} \frac{\ln G_\beta^{(\Lambda)}(\gamma)}{\ln d_L(x_0, y_0)} \geqslant \frac{\gamma^2}{2\pi} + O\{[\ln d_L(x_0, y_0)]^{-1}\} \tag{4.7}$$

which holds if *either* $\beta > \beta_c$ and $\lambda \sim \lambda_0$ (small activity) or $\beta > \bar{\beta}_c$ with no restrictions in $\lambda \in \mathscr{F}$.

From (4.6), (4.7), and (2.5) we can conclude that

$$\lim_{|x_0 - y_0| \to \infty} -\frac{1}{\beta} \frac{\ln G_\beta^{(\Lambda)}(\gamma)}{\ln d_L(x_0, y_0)} = \frac{\gamma^2}{2\pi} \tag{4.8}$$

i.e., the asymptotic behavior of the electrostatic potential between two external (noninteger) charges is not affected by the presence of the gas.

To prove (4.8), we perform renormalization group transformations both in the numerator and denominator of (4.2). In the numerator the two charges $\pm\gamma$ are initially in different blocks and after a number $n_0 = \log_L d_L(x_0, y_0)$ of iterations they are brought to the same block. Before this happens, i.e., $n \leqslant n_0$, we must discuss transformations of the type

$$\lambda_\eta^{(n)}(q) = \frac{1}{N_\eta^{(n)}} L^{-(\beta/4\pi)(q+\eta)^2} (\lambda^{(n-1)} * \cdots * \lambda^{(n-1)} * \lambda_\eta^{(n-1)})(q) \tag{4.9}$$

with $\eta = 0$, $\pm\gamma$, $\lambda_0^{(n)} \equiv \lambda^{(n)}$, and $\lambda_\eta^{(0)} \equiv \lambda$. The choice $\eta = 0$ corresponds to those blocks without external charges; $\eta = \pm\gamma$ corresponds to those blocks with charge $\pm\gamma$. When $N_\eta^{(n)} = [\lambda^{(n-1)} * \cdots * \lambda^{(n-1)} * \lambda^{(n-1)}](0)$, $N_\eta^{(n)} \equiv N^{(n)}$ (independent of η!).

For $n = n_0$ the two charges get into the same block, canceling each other. Further renormalization group transformation, involving this block, i.e., $n \geqslant n_0 + 1$, is described by

$$\hat{\lambda}_\gamma^{(n)}(q) = \frac{1}{N^{(n)}} L^{-(\beta/4\pi)q^2}(\lambda^{(n-1)} * \cdots * \lambda^{(n-1)} * \hat{\lambda}_\gamma^{(n-1)})(q) \tag{4.10}$$

with

$$\hat{\lambda}_\gamma^{(n_0+1)} = \frac{m_{\varphi_\beta}}{N^{(n_0+1)}} (\lambda^{(n_0)} * \cdots * \lambda_{+\gamma}^{(n_0)} * \lambda_{-\gamma}^{(n_0)}) \tag{4.11}$$

The main result of this section, inequality (4.7), is a consequence of the following lemmas.

Lemma 4.1. Let $\lambda \in \mathscr{F}$ satisfy the assumption of Theorem 3.1 for $\beta > \beta_c$ or Theorem 3.3 for $\beta > \bar{\beta}_c$. Then, for $n \leqslant n_0$,

$$\|\lambda_\eta^{(n)}\|_2 \leqslant C_1[\varphi_\beta(\eta)]^n \tag{4.12}$$

where C_1 is a n-independent constant.

Proof. For simplicity we consider $L^2 = 2$ only. By the Hausdorff–Young inequality

$$\begin{aligned} \|\lambda_\eta^{(n)}\|_2 &\leqslant \|\varphi_\beta(\cdot + \eta)(\lambda^{(n-1)} * \lambda_\eta^{(n-1)})\|_2 \\ &\leqslant \|\varphi_\beta(\cdot + \eta)\|_\infty \, \|\lambda^{(n-1)} * \lambda_\eta^{(n-1)}\|_2 \\ &\leqslant \varphi_\beta(\eta) \, \|\lambda^{(n-1)}\|_1 \, \|\lambda_\eta^{(n-1)}\|_2 \end{aligned} \tag{4.13}$$

Now $\|\lambda^{(n-1)}\|_1 \leqslant 1 + \delta^{n-1}$ from Theorem 3.1 or Theorem 3.3 and so

$$\|\lambda_\eta^{(n)}\|_2 \leqslant \varphi_\beta(\eta)(1 + \delta^{n-1}) \, \|\lambda_\eta^{(n-1)}\|_2 \tag{4.14}$$

where $0 < \delta < 1$ as given in Theorem 3.1 or Theorem 3.3.

Iterating this bound, we get

$$\|\lambda_\eta^{(n)}\|_2 \leqslant C_1[\varphi_\beta(\eta)]^n$$

Lemma 4.2. Let $\lambda \in \mathscr{F}$ satisfy the assumptions of Theorem 3.1 for $\beta > \beta_c$ or Theorem 3.3 for $\beta > \bar{\beta}_c$. Then for $n > n_0$

$$|\hat{\lambda}_\gamma^{(n)}(0)| \leqslant C_2 \, |\hat{\lambda}_\gamma^{(n_0+1)}(0)| \tag{4.15}$$

where C_2 is an n-independent constant.

Proof. We start from

$$|\hat{\lambda}_\gamma^{(n)}(0)| \leqslant |\hat{\lambda}_\gamma^{(n_0+1)}(0)| + \sum_{j=n_0+1}^{n-1} |\lambda_\gamma^{(j+1)}(0) - \lambda_\gamma^{(j)}(0)| \tag{4.16}$$

Since

$$\hat{\lambda}_\gamma^{(j+1)}(0) = \frac{1}{N^{(j+1)}} (\lambda^{(j)} * \hat{\lambda}_\gamma^{(j)})(0)$$

$$\begin{aligned} |\hat{\lambda}_\gamma^{(j+1)}(0) - \hat{\lambda}_\gamma^{(j)}(0)| &\leqslant |[(\lambda^{(j)} - \lambda_0) * \hat{\lambda}_\gamma^{(j)}](0)| \\ &= |(\chi^{(j)} * \hat{\lambda}_\gamma^{(j)})(0)| \leqslant \delta^j \, \|\hat{\lambda}_\gamma^{(j)}\|_\infty \end{aligned} \tag{4.17}$$

Now

$$\begin{aligned} \|\hat{\lambda}_\gamma^{(j)}\|_\infty &\leqslant \|\lambda^{(j-1)} * \hat{\lambda}_\gamma^{(j-1)}\|_\infty \\ &\leqslant \|\lambda^{(j-1)}\|_1 \, \|\hat{\lambda}_\gamma^{(j-1)}\|_\infty \leqslant (1 + \delta^{j-1}) \, \|\hat{\lambda}_\gamma^{(j-1)}\|_\infty \\ &\leqslant C_3 \, \|\hat{\lambda}_\gamma^{(n_0+1)}\|_\infty \leqslant C_3 \, \|\lambda_{+\gamma}^{(n_0)} * \lambda_{-\gamma}^{(n_0)}\|_\infty \\ &\leqslant C_3 \, \|\lambda_{+\gamma}^{(n_0)}\|_2 \, \|\lambda_{-\gamma}^{(n_0)}\|_2 = C_3 \, \|\lambda_\gamma^{(n_0)}\|_2^2 = C_3 \, |\hat{\lambda}_\gamma^{(n_0+1)}(0)| \end{aligned} \tag{4.18}$$

by the Hausdorf–Young inequality and Theorem 3.1 or Theorem 3.3.

From (4.16)–(4.18)

$$\begin{aligned} |\hat{\lambda}_\gamma^{(n)}(0)| &\leqslant \left(1 + C_3 \sum_{j=n_0+1}^{n-1} \delta^j\right) |\hat{\lambda}_\gamma^{(n_0+1)}(0)| \\ &\leqslant C_2 \, |\hat{\lambda}_\gamma^{(n_0+1)}(0)| \end{aligned} \tag{4.19}$$

We are now in a position to state and prove the main result of this section.

Theorem 4.3. Let λ satisfy the assumptions of Theorem 3.1 for $\beta > \beta_c$ or Theorem 3.3 for $\beta > \bar{\beta}_c$. Then, there exists a constant C independent of N such that

$$G_\beta^{(A_N)}(\gamma) \leqslant C[d_L(x_0, y_0)]^{-(\beta/2\pi)\gamma^2} \tag{4.20}$$

Proof. We first notice that

$$G_\beta^{(A_N)}(\gamma) = \hat{\lambda}_\gamma^{(N)}(0) \tag{4.21}$$

From Lemmas 4.2 and 4.1

$$\begin{aligned} |G_\beta^{(A_N)}(\gamma)| &\leqslant C_2 \, |\hat{\lambda}^{(n_0+1)}(0)| \\ &\leqslant C_1 C_2 [\varphi_\beta(\gamma)]^{2n_0} \quad \text{QED} \end{aligned} \tag{4.22}$$

5. BIFURCATION AND WEAK SCREENING

At $\beta=\beta_c$ a simple eigenvalue $\omega_1(\beta)$ of $l(\beta)=dr(0,\beta)$ crosses the unit circle and we can then use bifurcation theory to show that a stable line of nontrivial fixed points appear for $\beta \lesssim 8\pi$. Although the potential between two external fractional charges remains logarithmically confining, due to the hierarchical "disease" mentioned in the introduction, it becomes weaker than that for the trivial fixed point.

Following Crandall,[11] we first introduce $\hat{\mathscr{G}}=\{y\in\mathscr{G}\colon y(1)=0\}$ and define

$$F\colon \mathbb{R}\times\hat{\mathscr{G}}\times\mathbb{R}_+\to\mathscr{G}$$

$$F(t,y,\beta)=\begin{cases}(1/t)[r(t(e_1+y),\beta)-t(e_1+y)], & t\neq 0\\ [dr(0,\beta)-\mathbb{1}](e_1+y), & t=0\end{cases} \tag{5.1}$$

Notice that F is continuous in $t\in\mathbb{R}$.

We now show that the hypothesis of the implicit function theorem are satisfied, so that the equation

$$F(t,y,\beta)=0 \tag{5.2}$$

defines for t in a neighborhood I of 0, continuous function $g_t\in\hat{\mathscr{G}}$ and $\hat{\beta}_t\in\mathbb{R}_+$, with $g_0=0$, $\hat{\beta}_0=\beta_c$ satisfying $F(t,g_t,\hat{\beta}_t)=0$, $t\in I$. Therefore $\chi_t=t(e_1+g_t)$ is a nontrivial fixed point at temperature $\hat{\beta}_t$. Since $\hat{\beta}_t<\beta_c$, for $t\not\equiv 0$ we have a supercritical bifurcation and this will imply stability. These statements are made precise in the following result.

Theorem 5.1. Consider the function F given by (5.1); there exists then a neighborhood $V\subset\mathscr{G}\times\mathbb{R}_+$ of $(0,\beta_c)$, an open interval $I=(-a,a)\subset\mathbb{R}$, and continuous functions

$$\hat{\beta}\colon\begin{cases}I\to\mathbb{R}_+\\ t\to\beta_t\end{cases},\qquad g\colon\begin{cases}I\to\hat{\mathscr{G}}\\ t\to g_t\end{cases}$$

with

(a) $\hat{\beta}_0=\beta_c$, $\quad g_0=0$

(b) $F(t,t(e_1+g_t),\hat{\beta}_t)=0$, $\quad\forall t\in I$

Moreover, every solution $\chi\neq 0$ of $r(\chi,\beta)=0$ with $(\chi,\beta)\in V$ is of the type $(t(e_1+g_t),\hat{\beta}_t)$ for some $t\in I$. For $|t|$ sufficiently small the solution $\chi_t=t(e_1+g_t)$ is asymptotically stable.

Proof. As discussed in ref. 11, it is sufficient to check the assumptions of the implicit function theorem.

(a) $dr(\chi,\beta)$ and $(\partial dr/\partial\beta)(\chi,\beta)$ are both continuous in $\mathscr{G}\times\mathbb{R}_+$.

(b) The kernel of $dr(0, \beta_c) - \mathbb{1} = m(0, \beta_c)$, $\{x \in \mathscr{G}: dr(0, \beta_c)x = x\}$, has dimension 1 and the codimension of the range of $m(0, \beta_c)$ is one.

$$\text{(c)} \quad \left[\frac{\partial}{\partial \beta} dr(0, \beta_c)\right] e_1 = \alpha e_1 + y \tag{5.3}$$

with $\alpha \neq 0$ and $y \in \hat{\mathscr{G}}$.

Part (a) follows from an explicit computation for $dr(\chi, \beta)$. Part (b) follows from (3.19) and (3.20). Also, $dr(0, \beta)\, e_1 = L^{2-\beta/4\pi} e_1$ and part (c) follows.

In order to investigate the stability of fixed points χ_t, $t \in I$, we consider the expansion of $\hat{\beta}_t$ and g_t around $t = 0$ for $L^2 = 2$:

$$\begin{aligned} \hat{\beta}_t &= \beta_c \left[1 - \frac{1}{\ln 2} t^2 + O(t^4)\right] \\ g_t &= \frac{2^{-4}}{1 - 2^{-3}} \frac{e_2}{\sqrt{2}} t^2 + O(t^4) \end{aligned} \tag{5.4}$$

The above expansion corresponds to the usual ε expansion for $\lambda\phi^4$ theory in dimension $d = 4 - \varepsilon$. Therefore some $0 < b < a$

$$\hat{\beta}_t < \beta_c \qquad \text{for} \quad |t| < b \tag{5.5}$$

This means that we have a supercritical bifurcation. From the analysis in ref. 11 (Theorem 3, pp. 30, 31) we can conclude that the line χ_t of fixed points is stable for $|t| < b$. QED

Next we analyze the screening properties of the line of fixed points χ_t for $|t|$ sufficiently small.

Similarly to what was done in Section 4, for $n \leqslant n_0$ we are led to consider the transformation

$$\lambda_{t,\gamma}^{(n)} = \frac{m_{\varphi_{t,\gamma}}(\lambda_t * \cdots * \lambda_t * \lambda_{t,\gamma}^{(n-1)})}{(\lambda_t * \cdots * \lambda_t * \lambda_t)(0)} \tag{5.6}$$

where $\lambda_{t,\gamma}^{(0)} \equiv \lambda_t$ and $\lambda_t = \lambda_0 + \chi_t = \lambda_0 + t(e_1 + g_t)$. As before, m_f is the multiplication operator by the function f, and

$$\varphi_{t,\gamma}(q) = L^{-(\hat{\beta}_t/4\pi)(q+\gamma)^2} \tag{5.7}$$

g_t and $\hat{\beta}_t$ are given by Theorem 5.1.

The map $\lambda_{t,\gamma}^{(n)} \to \lambda_{t,\gamma}^{(n+1)}$ defines a linear operator $M_{t,\gamma}: l_1(\mathbb{Z}) \to l_1(\mathbb{Z})$:

$$\lambda \to \lambda' = M_{t,\gamma} \lambda \tag{5.8}$$

Up to order t^2 (and for $L^2=2$) the matrix elements of $M_{t,\gamma}$ are given by

$$M_{t,\gamma}(q,q')=\frac{\varphi_{t,\gamma}(q)}{1+t^2+O(t^6)}\left[\delta_{q,q'}+\frac{1}{\sqrt{2}}(\delta_{q,q'+1}+\delta_{q,q'-1})t+O(t^3)\right] \tag{5.9}$$

and its eigenvalues $\mu_i(t,\gamma)$, $i\in\mathbb{Z}$, and eigenvectors $f_i(t,\gamma)$, $i\in\mathbb{Z}$, can be computed explicitly [the numbering $f_i(t,\gamma)$ is chosen so that $f_i(0,\gamma)(q)=\delta_{q,i}\equiv f_i(q)$]:

$$\begin{aligned}\mu_i(t,\gamma)&=\varphi_{t,\gamma}(i)\left\{1+\left[\frac{1}{2}(c_{i,i+1}+c_{i,i-1})-1\right]t^2\right\}+O(t^4)\\ f_i(t,\gamma)&=f_i+\frac{1}{\sqrt{2}}(c_{i,i+1}f_{i+1}+c_{i,i-1}f_{i-1})t+O(t^3)\end{aligned} \tag{5.10}$$

where

$$c_{ij}\equiv c_{ij}(t,\gamma)=\frac{\varphi_{t,\gamma}(j)}{\varphi_{t,\gamma}(i)-\varphi_{t,\gamma}(j)}$$

Therefore the eigenvalue with largest absolute value is

$$\mu_0(t,\gamma)=\varphi_{t,\gamma}(0)\left\{1+\left[\frac{1}{2}(c_{0,1}+c_{0,-1})-1\right]t^2\right\}\equiv\varphi_{t,\gamma}(0)(1+K_\gamma t^2) \tag{5.11}$$

where $K_\gamma\geqslant 0$, $K_0=0$, so that $\mu_0(t,0)=1$.

The $\{f_i(t,\gamma), i\in\mathbb{Z}\}$ form a basis in $l_1(\mathbb{Z})$ and writing $\lambda_t=\sum_i\alpha_i f_i(t,\gamma)$, we obtain

$$\lambda_{t,\gamma}^{(n)}=\sum_{i\in\mathbb{Z}}\alpha_i[\mu_i(t,\gamma)]^n f_i(t,\gamma) \tag{5.12}$$

For $n\geqslant n_0+1$ we are led to consider the map

$$\begin{aligned}\hat{\lambda}_{t,\gamma}^{(n+1)}&=\frac{m_{\varphi_{t,0}}(\lambda_t*\cdots*\lambda_t*\hat{\lambda}_{t,\gamma}^{(n)})}{(\lambda_t*\cdots*\lambda_t*\lambda_t)(0)}\\ \hat{\lambda}_{t,\gamma}^{(n_0+1)}&=\frac{m_{\varphi_{t,0}}(\lambda_t*\cdots*\lambda_{t,+\gamma}^{(n_0)}*\lambda_{t,-\gamma}^{(n_0)})}{(\lambda_t*\cdots*\lambda_t*\lambda_t)(0)}\end{aligned} \tag{5.13}$$

Therefore the leading contribution to $G_{\beta_t}^{(\Lambda_N)}(\gamma)$ is given by $[\mu_0(t,\gamma)]^{2n_0}$, i.e.,

$$\frac{-\ln G_{\beta_t}^{(\Lambda_N)}(\gamma)}{\ln d_{\sqrt{2}}(x_0,y_0)}=\left(\frac{\hat{\beta}_t\gamma^2}{2\pi}-\frac{2K_\gamma t^2}{\ln 2}\right)(1+\xi_{\Lambda_N}^{n_0}) \tag{5.14}$$

where $|\zeta_{\Lambda_N}|\leqslant\varepsilon<1$, $\forall n_0$, N, and γ, which establishes our result.

With little extra effort one could establish along the same lines of Section 4 weak screening not just for the fixed point, but also for λ sufficiently close to λ_t.

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